**Journal of the Operational Research Society (2000) 51**, 341–349  0160-5682/000 $15.00

http://www.stockton-press.co.uk/jors

# A fuzzy programming method for deriving priorities in the analytic hierarchy process

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The estimation of the priorities from pairwise comparison matrices is the major constituent of the Analytic Hierarchy Process (AHP). The priority vector can be derived from these matrices using different techniques, as the most commonly used are the Eigenvector Method (EVM) and the Logarithmic Least Squares Method (LLSM). In this paper a new Fuzzy Programming Method (FPM) is proposed, based on geometrical representation of the prioritisation process. This method transforms the prioritisation problem into a fuzzy programming problem that can easily be solved as a standard linear programme. The FPM is compared with the main existing prioritisation methods in order to evaluate its performance. It is shown that it possesses some attractive properties and could be used as an alternative to the known prioritisation methods, especially when the preferences of the decision-maker are strongly inconsistent.

**Keywords:** AHP; prioritisation methods; fuzzy programming

## Introduction

The Analytic Hierarchy Process (AHP) is one of the most popular methods of the Multi-Criteria Decision Making (MCDM). In the AHP the weights of criteria and the scores of alternatives are not explicitly distinguished, as in the direct assessment MCDM methods. The weights are derived from judgement matrices of pairwise comparisons of the importance of the criteria and the scores are generated from pairwise comparisons of the alternatives with respect to a particular criterion at the upper level. The elicitation process for both weights and scores is identical, that is why they are often called *priorities*.

The estimation of priorities from pairwise comparison matrices is the major constituent of the AHP. The priority vector can be derived from the comparison matrices using different techniques. The traditional method, proposed by Saaty[1] is the Eigenvector Method (EVM). Saaty proves that the principal eigenvector of the comparison matrix can be used as a priority vector for consistent and inconsistent preferences.

Most other methods for deriving priorities in the AHP are based on some optimisation approach. They introduce an objective function, which measures the distance between an ‘ideal’ solution and the actual one. Then the problem of priorities derivation is to minimise this objective function subject to some additional constraints. Such optimisation methods are the Direct Least Squares Method (DLSM), minimising the Euclidean distance from the given comparison matrix under additive normalisation constraints and the Weighted Least Squares Method (WLSM), using a modified Euclidean norm as an objective function.[2] The Logarithmic Least Squares Method (LLSM) of Crawford and Williams[3] makes use of the multiplicative properties of the pairwise comparison matrices and applies an optimisation procedure to minimise a logarithmic objective function, subject to multiplicative constraints. This method gives an explicit solution, which is rather simple and convenient from computational point of view.

Some authors use a goal programming approach to solve the prioritisation problem. Between them we can reference the logarithmic Goal Programming Method (GPM) proposed by Bryson,[4] which minimises a linear logarithmic function subject to some linear constraints. Therefore far the most appealing techniques for prioritisation in the AHP are the EVM and the LLSM. Many researchers have performed comparisons between these methods in order to evaluate their performance and to favour one of them, but their conclusions are often contradictory. Some authors such as Crawford and Williams,[3] Barzilai[5] and Zahedi[6] assert that the LLSM overperforms the EVM. Other researchers claim that the EVM is inferior to the LLSM.[7,8] Takeda *et al*[9] apply more comparison criteria and test the main prioritisation methods with a great number of randomly generated pairwise matrices. Their findings are that the LLSM is superior to the EVM in some cases and equal in others.

An excellent comparison analysis between the commonly used methods for deriving priorities is given

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in Golany and Kress.[10] They conclude that there is no prioritisation method that is superior to the other ones in all cases. All methods have their advantages and drawbacks and the choice of the prioritisation method should be dictated by the objective of the analysis. This conclusion justifies our study in this area and our efforts to develop and test a new approach for prioritisation in the AHP.

The main objective of this paper is to present new Fuzzy Programming Method (FPM) for priorities derivation from pairwise comparison matrices. The FPM is based on a geometrical representation of the prioritisation process as an intersection of fuzzy hyperlines and determines the values of the priorities, corresponding to the point with the highest measure of intersection. Therefore the FPM can be considered as an optimisation approach for deriving the best priority vector. Using such an approach, the prioritisation problem is reduced to a fuzzy programming problem that can easily be solved as a standard linear program.

The FPM is further compared with the main prioritisation methods mentioned above. The comparison itself is of secondary importance to this paper and serves only to justify the proposed method and to evaluate its properties. It is shown that the new FPM has some attractive properties, such as a natural consistency indicator, simplicity of the computation algorithm, good precision and rank preservation, and can be used as an alternative to the known prioritisation methods.

## The prioritisation methods in the AHP

In the AHP the decision problem is structured hierarchically at different levels, each level consisting of a finite number of elements. The priorities represent the relative importance of the decision elements at that level. For all levels of the hierarchy the prioritisation of the elements is carried out with respect to the elements of the upper level. The elicitation of the priorities at a given level is performed by pairwise comparisons. The pairwise comparison in the AHP assumes that the decision-maker can compare any two elements and to provide a numerical value of the ratio of their relative importance.

Let us consider prioritisation of $n$ elements $E_1, E_2, \ldots, E_n$ at the same level of hierarchy. Comparing any two elements $E_i$ and $E_j$, the decision-maker assigns the value $a_{ij}$, which represents a judgement concerning the relative importance of preference of decision element $E_i$ over $E_j$. If the element $E_i$ is preferred to $E_j$ then $a_{ij} > 1$. Correspondingly, the reciprocal property $a_{ji} = 1/a_{ij}$ for $j = 1, 2, \ldots, n, i = 1, 2, \ldots, n$ always holds.

Each set of comparisons for a level with $n$ elements requires $n(n-1)/2$ judgements. In such a way a positive reciprocal matrix of pairwise comparisons $A = \{a_{ij}\} \in \Re^{nxn}$ is constructed. Then a priority vector $w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ may be derived from this matrix. The set of $n$ relative priorities is normalised to sum of one, therefore the number of independent normalised priorities is $(n-1)$. When the decision-maker is perfectly consistent in his judgements then all elements $a_{ij}$ have perfect values $a_{ij} = w_i/w_j$. In this case we have $a_{ij} = a_{ik}a_{kj}$ for all $i, j, k = 1, 2, \ldots, n$. Then the pairwise comparison matrix $A$ is said to be consistent and can be represented as $A_c = \{w_i/w_j\}$. The consistent priorities are unique and readily available by taking the elements in any column of the comparison matrix $A_c$ and then dividing each of them by the sum of all elements of the column.

However, the decision-maker's evaluations $a_{ij}$ are frequently not perfect, they are only estimations of the exact ratios $w_i/w_j$. Such inconsistent judgements are more common and then $A$ is an inconsistent matrix, which can be considered as a perturbation of the consistent one $A_c$. The inconsistent priorities are not unique and should be derived using some estimation technique. We will briefly describe the main methods for priorities derivation from pairwise comparison matrices that are used in the comparison analysis to follow.

### *Eigenvector method (EVM)*

This is the original Saaty's approach to derive the priorities in the AHP. The EVM is based on the fact that small perturbations of the elements $a_{ij}$ from the perfect ratios $w_i/w_j$ lead to small perturbations of the eigenvalues of the comparison matrix $A$ around the eigenvalues of the consistent one $A_c$. Using the Frobenius Theorem, Saaty[1] proves that the principal eigenvector of $A$ can be used as the desired priority vector. So the EVM is based on solving the equation:

$$Aw = \lambda_{\max} w, \lambda_{\max} \geq n. \tag{1}$$

For small deviations around the perfect evaluations this approach gives reasonably good approximation of the priority vector, but when the inconsistency of the decision-maker's preferences is large, then the solutions are not satisfactory.

### *Least squares methods*

The Direct Least Squares Method (DLSM), proposed by Chu *et al*[2] is based on the assumption that the elements of the priority vector $w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ should best satisfy the property $a_{ij} \approx w_i/w_j$. Therefore the priorities assessment is formulated as a constrained optimisation problem:

$$\min \sum_i \sum_j (a_{ij} - w_i/w_j)^2 \tag{2}$$

subject to

$$\sum_{i=1}^{n} w_i = 1, w_i > 0, \qquad i = 1, 2, \ldots, n. \tag{3}$$

The above nonlinear optimisation problem has no special tractable form and generally has multiple solutions.[2,10] In order to eliminate the drawbacks of the DLSM, Chu *et al*[2] modify the objective function (2) in the following form:

$$\min \sum_i \sum_j (w_i - a_{ij} w_j)^2 \tag{4}$$

The Weighted Least Squares Method (WLSM) consists in minimisation of (4), subject to the additive normalising and non-negative constraints given by (3).

The WLSM reduces the solution of the above optimisation problem to a system of linear equations that can easily be solved. It is shown in Blankmeyer[11] that the WLSM provides a unique solution, unlike the DLSM. That is why the WLSM is chosen for the comparison analysis in this paper.

*Logarithmic least squares method (LLSM)*

The LLSM, also known as Geometric Mean Method (GMM) is a variation of the above least square methods and it is widely used due to its simplicity. The LLSM minimises the objective function

$$\sum_i \sum_j (\ln\ a_{ij} - \ln\ w_i + \ln\ w_j)^2 \tag{5}$$

subject to the multiplicative normalising constraints

$$\prod_{i=1}^{n} w_i = 1, w_i > 0, \qquad i = 1, 2, \ldots, n. \tag{6}$$

Crawford and Williams[3] have proved the validity of this method. They have shown, that the solution is unique and can be found as the geometric mean of the columns of $A$,

$$w_i = \prod_{j=1}^{n} (a_{ij})^{1/n}, \qquad i = 1, 2, \ldots, n. \tag{7}$$

*Goal programming method (GPM)*

This method, proposed by Bryson[4] uses the consideration, that the priorities are desired to satisfy the equalities

$$\left(\frac{w_i}{w_j}\right)\left(\frac{\delta_{ij}^{+}}{\delta_{ij}^{-}}\right) = a_{ij}, \quad i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, n,\ j > i, \tag{8}$$

where $\delta_{ij}^{+} \geq 1$ and $\delta_{ij}^{-} \geq 1$ are additional deviation variables, which cannot both be greater than 1. The priorities are obtained as solutions of the following linear goal programming problem:

$$\min \sum_{i=1}^{n} \sum_{j>1}^{n} (\log \delta_{ij}^{+} + \log \delta_{ij}^{-}) \tag{9}$$

subject to

$$\log\ w_i - \log\ w_j + \log\ \delta_{ij}^{+} - \log\ \delta_{ij}^{-} = \log\ a_{ij}, \quad i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, n,\ j > i, \tag{10}$$

where all $\log\ \delta_{ij}^{+}$ and $\log\ \delta_{ij}^{-}$ are non-negative.

## Geometrical representation of the priorities derivation problem

Generally, when we have $n$ priorities, we need $n(n-1)/2$ evaluation elements $a_{ij}$, each one of them being estimation of the real ratio $w_i/w_j$. If the reciprocal matrix of pairwise comparisons $A = \{a_{ij}\} \in \Re^{nxn}$ is consistent, then $a_{ij} w_j - w_i = 0$ for all $i = 1, 2, \ldots, n,\ j = 2, 3, \ldots, n,\ j > i$. This can be represented as a system of linear equations:

$$Rw = 0, \tag{11}$$

where the matrix $R \in \Re^{mxn}, m = n(n-1)/2$ is obtained from the elements of the matrix $A$. Therefore we can formulate the problem of priorities assessment in the following way: Find a positive priority vector $w$ that satisfies (11) and the normalisation equation

$$\sum_{i=1}^{n} w_i = 1,\ w_i > 0, \qquad i = 1, 2, \ldots, n. \tag{12}$$

Each row of (11) $R_j w = 0$ defines a *hyperplane* in the $n$-dimensional priority space. We shall denote the $j$th hyperplane as

$$H_j(w) = \{w | R_j w = 0\}, \qquad j = 1, 2, \ldots, m. \tag{13}$$

Since the solution of the prioritisation problem must lie on the simplex hyperplane $H_0(w) = \{w | w_1 + w_2 + \cdots + w_n = 1\}$, the intersections of the hyperplanes $H_j(w)$ with $H_0(w)$ should be considered.

The intersection between each hyperplane $H_j(w)$ and $H_0(w)$ is defined as a *hyperline* $L_j(w)$

$$L_j(w) = H_j(w) \cap H_0(w), \qquad j = 1, 2, \ldots, m. \tag{14}$$

Then the exact solution of the problem (if it exists) can be represented as a *common intersection*$P(w)$ of all hyperlines $L_j(w)$ on $H_0(w)$:

$$P(w) = L_1(w) \cap L_2(w) \cap \cdots \cap L_m(w). \tag{15}$$

For the perfectly consistent case when $a_{ij} = a_{ik} a_{kj}, \forall i, k, j$, the common intersection $P(w)$ of all hyperlines $L_j(w)$ on the simplex hyperplane is not empty and contains only one point, which gives the exact solution of (11) and (12), $P(w) = \{w^*\}, w^* = (w_1^*, w_2^*, \ldots, w_n^*)^T$. In the inconsistent case a common intersection point of all hyperlines on $H_0(w)$ does not exist, so the intersection set is empty, $P(w) = 0$. Then it is desirable to find such values of $w$, so that (11) is approximately satisfied, that is, $Rw \approx 0$.

Before describing our approach to find such approximate solution, let us illustrate the geometrical representation of the prioritisation problem, using the following example.

*Example 1. Three-dimensional prioritisation problem*

Consider a three-dimensional prioritisation problem with unknown priorities $w_1, w_2$ and $w_3$. The pairwise comparison matrix $A$ is

$$A = \begin{bmatrix} 1 & a_{12} & a_{13} \\ 1/a_{12} & 1 & a_{23} \\ 1/a_{13} & 1/a_{23} & 1 \end{bmatrix} \qquad (16)$$

We have to evaluate the three ratios, $a_{12}, a_{13}$ and $a_{23}$. Then we can represent (11) as a system of three equations:

$$w_1 a_{12} w_2 = 0; \qquad w_1 - a_{13} w_3 = 0; \qquad w_2 - a_{23} w_3 = 0. \qquad (17)$$

These equations define three planes in the 3-dimensional priority space, which intersect the simplex plane $\{w_1 + w_2 + w_3 = 1, w_i > 0, i = 1, 2, 3\}$. The three intersection lines might intersect each other at one single point or at three different points on the simplex plane. For the perfectly consistent case we have $a_{13} = a_{12} a_{23}$ and an exact solution of the above problem exists, represented by the common intersection point of the three intersection lines on the simplex plane (Figure 1).

Let the comparison elements be $a_{12} = 2, a_{13} = 6$ and $a_{23} = 3$. For these values of the elements the corresponding matrix $A$ is consistent and the exact solution is given by $w^* = (0.6, 0.3, 0.1)^T$.

If the matrix $A$ is inconsistent, an exact solution, satisfying simultaneously all three equations in (17) does not exist, hence the intersection lines have no common intersection point. Approximate solutions of the problem could be found in the triangular area or at its vertices, shown on Figure 2 as a shaded area. The EVM and all least square

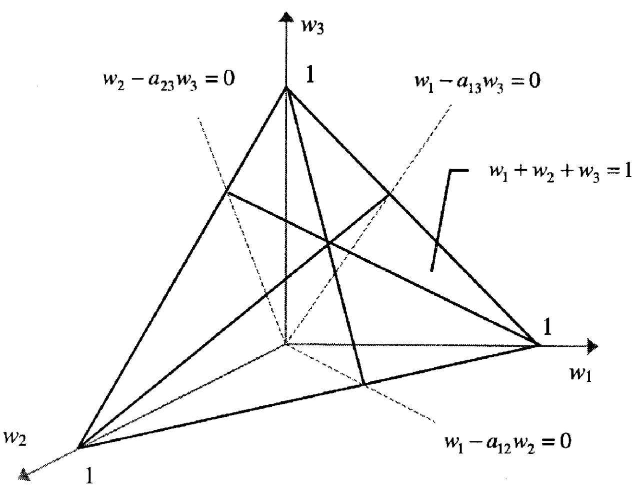


**Figure 1** Graph of consistent preferences.

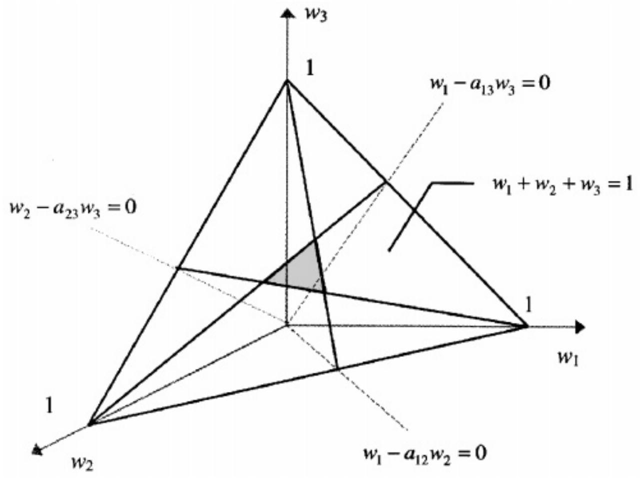


**Figure 2** Graph of inconsistent preferences.

methods give solutions inside this triangular area, while the GPM provides solution located at one of the vertices of this simplex.

## A fuzzy approach to priorities derivation

Our approach for determination of the priority vector is based on the above geometrical representation of the prioritisation problem. The main idea behind this approach is to represent the hyperlines $L_j(w)$ as fuzzy lines and to find the solution of the approximate priority assessment problem, as an intersection point of these fuzzy lines.

The following notation and definitions are used to describe the fuzzy method for priority assessment. A *fuzzy set* $\tilde{A}$ is defined as a set of ordered pairs $\tilde{A} = \{(x, m_{\tilde{A}}(x)) | x \in X\}$, where $m_{\tilde{A}}(x)$ is a *membership function*, which maps each element $x$ to a real value in [0,1]. The crisp sets of the elements $x \in X$ that belong to the fuzzy set $\tilde{A}$ to degree of $\alpha$ are called *$\alpha$-level sets* (or simply $\alpha$-cuts) of $\tilde{A}$, and are defined as $A(\alpha) = \{x \in X | m_{\tilde{A}}(x) \geq \alpha\}$. Using this concept each fuzzy set $\tilde{A}$ can be represented as a sequence of sets $A(\alpha_i)$, $0 = \alpha_1 < \alpha_2 < \cdots \alpha_k = 1$. For $\alpha = 0$ the corresponding $\alpha$-level $A(0)$ represents the *support* of the fuzzy set $\tilde{A}$, while $A(1)$ is the *core* of $\tilde{A}$.

The core of a normal fuzzy set is nonnull subset. In this paper we will consider a specific form of normal fuzzy sets, which are called *fuzzy numbers*. The normal fuzzy set $\tilde{N}$ is a triangular fuzzy number if it has a piecewise continuous membership function $m_{\tilde{N}}(x)$:[12]

1. A continuous mapping from $\Re$ to the closed interval [0,1];
2. $m_{\tilde{N}}(x) = 0$ for all $x \in [-\infty, a]$ and for all $x \in [c, +\infty]$;
3. Strictly increasing on $[a, b]$ and strictly decreasing on $[b, c]$;
4. $m_{\tilde{N}}(x) = 1$ for $x = b$.

where $a < b < c$ are real numbers.

The $\alpha$-level sets of the fuzzy number $\tilde{N}$ are closed intervals, whereas $\tilde{N}(\alpha_k) \subset \tilde{N}(\alpha_{k1} \subset \cdots \subset \tilde{N}(0)$.

### *A fuzzy intersection hyperline*

The intersection between the hyperplanes $H_j(w)$ and $H_0(w)$ is defined as a *fuzzy intersection hyperline* $\tilde{L}_j$ on $H_0(w)$, which is a triangular fuzzy number

$$\tilde{L}_j = \{(w, m_{\tilde{L}_j}(w)) | w \in \Re^{n+}\},$$

characterised by its linear membership function $m_{\tilde{L}_j}(w)$

$$m_{\tilde{L}_j}(w) = \begin{cases} 0, & R_j w < -d_j^- \\ 1 + \dfrac{R_j w}{d_j^-}, & R_j w \in [-d_j^- 0] \\ 1, & R_j w = 0 \\ 1 - \dfrac{R_j w}{d_j^+}, & R_j w \in [0, d_j^+] \\ 0, & R_j w > d_j^+ \\ w_1 + w_2 + \cdots w_n = 1 \end{cases} \tag{18}$$

This triangular membership function is a modification of that initially suggested by Zimmermann[13] for solving of fuzzy linear programming problems. The membership function $m_{\tilde{L}_j}(w)$ equals 1 when the corresponding crisp equation of the plane $R_j w = 0$ is fully satisfied. This function is equal to 0 when the plane equation is strongly violated and it takes values between 0 and 1 if it is approximately satisfied (note that the additive normalisation constraint always holds). The membership function is linearly increasing over the interval $[-d_j^-, 0]$ and linearly decreasing over the interval $[0, d_j^+]$ The values of the left and right *tolerance parameters* $d_j^-$ and $d_j^+$ represent the admissible interval of approximate satisfaction of the crisp equality $R_j w = 0$ on the simplex hyperplane.

### *An intersection region of the fuzzy hyperlines*

Let $\tilde{L}_1$ and $\tilde{L}_2$ be two fuzzy hyperlines on the simplex hyperplane, which cores are denoted by $l_1(w)$ and $l_2(w)$. In general, these fuzzy hyperlines intersect at a *fuzzy region* $\tilde{P} = \tilde{L}_1 \cap \tilde{L}_2$, whose membership function can be defined as in Buckley[14]:

$$m_{\tilde{P}}(w) = \min(m_{\tilde{L}_1}(w), m_{\tilde{L}_2}(w)). \tag{19}$$

The intersection of two fuzzy lines on the simplex plane $H_0(w)$ in the three-dimensional priority case is shown on Figure 3. If the cores of the hyperlines $l_1(w)$ and $l_2(w)$ intersect, as in Figure 3(a) the fuzzy intersection region is not empty and contains many points with different degrees of membership in this region. It should be noted that the intersection region $\tilde{P}$ of two hyperlines with parallel cores $l_1(w)$ and $l_2(w)$ could also be nonempty if the spreads of these lines, given by their parameters $d_j^-$ and $d_j^+$ are large enough as shown in Figure 3(b). In this case the fuzzy intersection region is itself a fuzzy hyperline.

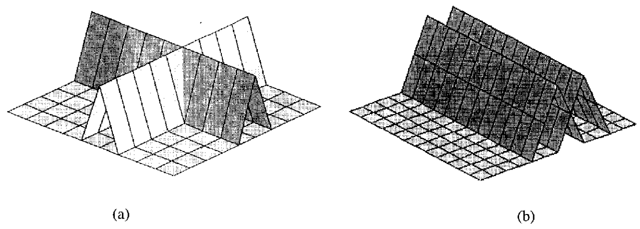


**Figure 3** Intersection of two (a) perpendicular, and (b) parallel fuzzy lines on the simplex plane.

Similarly, the intersection of $m$ fuzzy lines is defined as a fuzzy region $\tilde{P} = \tilde{L}_1 \cap \tilde{L}_2 \cap \cdots \cap \tilde{L}_m$, which is a fuzzy set with a membership function

$$m_{\tilde{P}}(w) = \min(m_{\tilde{L}_1}(w), m_{\tilde{L}_2}(w), \ldots, m_{\tilde{L}_m}(w)). \tag{20}$$

If the spreads of the membership functions of the fuzzy lines, given by the left and right tolerance parameters $d_j^-$ and $d_j^+$ are large enough, then the fuzzy set $\tilde{P}$ is not empty.

**Theorem 1** *The intersection region of m fuzzy lines is a convex fuzzy set.*

*Proof* Consider a non-empty fuzzy intersection region $\tilde{P}$. Let $w^1$ and $w^2$ be two points of the $\alpha$-level set of $P(\alpha)$, $w^1 \in P(\alpha)$, $w^2 \in P(\alpha)$. Then

$$m_{\tilde{L}_j}(w^1) \geq \alpha \quad \text{and} \quad m_{\tilde{L}_j}(w^2) \geq \alpha \qquad \text{for all } j = 1, 2, \ldots, m.$$

Consider a new point $w = \beta w^1 + (1 - \beta) w^2, 0 < \beta < 1$. Since all $m_{\tilde{L}_j}(w)$ are defined in (18) as convex functions, we have

$$m_{\tilde{L}_j}(w) = m_{\tilde{L}_j}(\beta w^1 + (1 - \beta) w^2) \geq \min(m_{\tilde{L}_j}(w^1), m_{\tilde{L}_j}(w^2)) \geq \alpha \qquad \text{for all } j = 1, 2, \ldots, m.$$

Therefore

$$m_{\tilde{P}}(w) = \min(m_{\tilde{L}_1}(w), m_{\tilde{L}_2}(w), \ldots, m_{\tilde{L}_m}(w)) \geq \alpha.$$

It follows that the new point $w \in P(\alpha)$, hence the intersection region $\tilde{P}$ is a convex fuzzy set. ■

### *Measure of intersection of the fuzzy hyperlines*

The *measure of intersection* of two fuzzy hyperlines $\tilde{L}_1$ and $\tilde{L}_2$ is defined by Buckley[14] as

$$\mu = \max_{\Re^n}\{\min(m_{\tilde{L}_1}(w), m_{\tilde{L}_2}(w))\} \tag{21}$$

For a given point $w \in \Re^n$ on $H_0(w)$ the measure $\mu$ represents the height of the intersection region of the fuzzy lines $\tilde{L}_1$ and $\tilde{L}_2$. If $\tilde{P}$ is an empty set then $\mu = 0$. If the fuzzy intersection region is a fuzzy hyperline then $0 < \mu \leq 1$, and $\mu$ is equal to 1 at the intersection point between the core lines.

Since the fuzzy intersection region is a convex set we can define the measure of intersection $\mu$ of $m$ fuzzy lines by analogy as

$$\mu = \max_{\Re^n}\{\min(m_{\tilde{L}_1}(w), m_{\tilde{L}_2}(w), \ldots, m_{\tilde{L}_m}(w))\} \qquad (22)$$

Therefore the problem of priorities assessment can be viewed as an optimisation task for selection of a priority vector $w^*$ in the intersection region, which maximises the measure of intersection.

It should be noted that (22) is similar to the *maximising decision* in the decision making in fuzzy environment with fuzzy goals and fuzzy constraints, proposed by Bellman and Zadeh.[15] Since the membership function and the additive normalising constraint are linear, our prioritisation problem could be seen as the fuzzy linear programming problem, studied by Zimmermann.[13]

As in the fuzzy linear programming we can transform (22), using (18) in the form

$$\mu = \max_{w}\left[\min\left\{\left(1-\frac{R_1 w}{d_1^+}\right), \left(1+\frac{R_1 w}{d_1^-}\right), \ldots, \times\left(1-\frac{R_m w}{d_m^+}\right), \left(1+\frac{R_m w}{d_m}\right)\right\}\right] \qquad (23)$$

where the normalisation condition $\sum_{i=1}^{n} w_i = 1$ is satisfied.

The formulation of the max-min problem given by (23) is equivalent to the following linear program:

$$\max\ \mu$$

subject to

$$\begin{aligned} &\mu d_j^+ + R_j w \le d_j^+, \\ &\mu d_j^- - R_j w \le d_j^- \qquad j = 1, 2, \ldots, m,\ 1 \ge \mu \ge 0; \\ &\sum_{i=1}^{n} w_i = 1,\ w_i > 0, \qquad i = 1, 2, \ldots, n. \end{aligned} \qquad (24)$$

The FPM transforms the prioritisation problem into the linear program (24) that can easily be solved by the standard simplex method. It should be noted that the above equations could be derived without geometrical representations, if the initial prioritisation problem were directly formulated as a fuzzy programming problem. The geometrical considerations however will give the reader a better understanding of the main idea of the proposed prioritisation method.

The measure of intersection $\mu$ is a natural *consistency index* of the FPM. Its value however depends on the tolerance parameters. For the practical implementation of the FPM it is reasonable all these parameters to be set equal.

## Comparison analysis of the fuzzy programming method

Let us consider again the problem of elicitation of three priorities, described in Example 1 from the previous section. In order to illustrate the solution of this problem by the proposed FPM, we can set $d_j^+ = d_j^- = 1$, $j = 1, 2, 3$, and to formulate the corresponding linear program, (24) as:

$$\max\ \mu$$

subject to

$$\begin{aligned} &\mu + (w_1 - a_{12}w_2) \le 1; \\ &\mu - (w_1 - a_{12}w_2) \le 1; \\ &\mu + (w_1 - a_{13}w_3) \le 1; \\ &\mu - (w_1 - a_{13}w_3) \le 1; \\ &\mu + (w_2 - a_{23}w_3) \le 1; \\ &\mu - (w_2 - a_{23}w_3) \le 1; \\ &w_1 + w_2 + w_3 = 1, w_i > 0, \qquad i = 1, 2, 3. \end{aligned}$$

Let the comparison elements be $a_{12} = 2, a_{13} = 5$ and $a_{23} = 3$. For these values of the elements the corresponding matrix $A$, (16) is inconsistent, so only approximate priorities can be obtained.

The values of the priorities derived by the FPM for this example are given in Table 1 and are compared with the solutions of the Saaty's EVM, LLSM, WLSM and GPM. The comparison is carried out, using the Euclidean distance criterion[10]:

$$E = \left[\sum_i \sum_j \left(a_{ij} - \frac{w_i}{w_j}\right)^2\right]^{1/2} \qquad (25)$$

The largest eigenvalue in the Saaty's EVM solution is $\lambda_{\max} = 3.004$, which suggests that the comparison matrix is very slightly inconsistent. The Saaty's consistency index is $CI = 0.002$, and the consistency ratio is $CR = 0.003 < 0.1$ (0.1 is the maximum allowable CR for the EVM). The consistency index $\mu$ of the FPM for this problem is also very close to unity, $\mu = 0.972$.

From Table 1 it is seen that the FPM gives the best approximation to the initial preferences, followed by the WLSM. It should be noted that the EVM and LLSM derive equal priorities. This result is not surprising, since in the three-dimensional case the normalised geometric mean and the eigenvector are identical, as it is shown in Crawford and Williams.[3]

The comparison matrix used in this example is very close to the consistent one (when $a_{13} = 6$), that is why the priorities derived by all methods are very close. In order to perform better comparative analysis of the proposed FPM with the other methods for prioritisation, we have

**Table 1** Comparison of the prioritisation methods

| *Solution method* | $w_1$ | $w_2$ | $w_3$ | $\frac{w_1}{w_2}$ | $\frac{w_1}{w_3}$ | $\frac{w_2}{w_3}$ | $E$ |
|---|---|---|---|---|---|---|---|
| FPM | 0.583 | 0.306 | 0.111 | 1.909 | 5.250 | 2.750 | 0.365 |
| EVM | 0.582 | 0.309 | 0.109 | 1.882 | 5.313 | 2.823 | 0.379 |
| LLSM | 0.582 | 0.309 | 0.109 | 1.882 | 5.313 | 2.823 | 0.379 |
| WLSM | 0.585 | 0.302 | 0.113 | 1.937 | 5.189 | 2.679 | 0.377 |
| GPM | 0.588 | 0.294 | 0.118 | 2.000 | 5.000 | 2.500 | 0.500 |

generated a number of three-dimensional comparison matrices, (26)

$$A = \begin{bmatrix} 1 & 2 & a_{13} \\ \frac{1}{2} & 1 & 3 \\ 1/a_{13} & \frac{1}{3} & 1 \end{bmatrix} \quad (26)$$

All matrices have fixed values of $a_{12} = 2$ and $a_{23} = 3$, only the third element $a_{13}$ varies in a wide range between 0.01 and 100, therefore permitting high degrees of inconsistency.

The value of the criterion $E$, (25) as a function of the parameter $a_{13}$ are given in Table 2. In the consistent case all prioritisation methods give the same solution and $E$ equals zero. For very great values of the parameter $a_{13}$, the EVM and the LLSM are overperformed by all other methods. When $a_{13} < 6$, either the EVM (LLSM) or the FPM provide the best solution. For $a_{13} > 6$ the FPM always gives the best approximation to the initial preferences, while the WLSM ranks second. It should be noted that the FPM is the best performer when the matrix $A$ is highly inconsistent.

The consistency index $\mu$ of the FPM and the Saaty's ratio CR are graphically represented on Figure 4 as a function of the parameter $a_{13}$. It is seen that the FPM is more robust than the EVM, therefore it can be used for deriving priorities from very inconsistent comparison matrices.

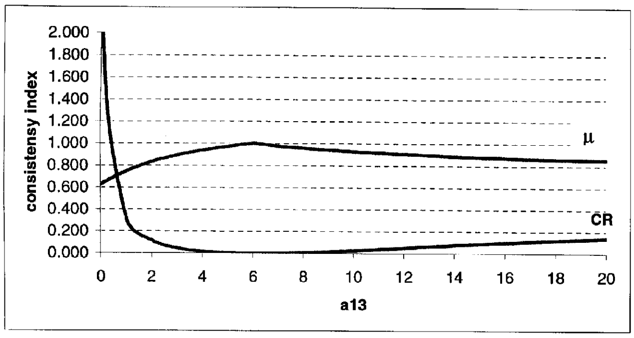


**Figure 4** Consistency index of the FPM and the EVM.

*Rank preservation*

Another important property of the prioritisation methods is their ability to preserve the ordinal preferences, which are implicitly expressed by the elements of the comparison matrix A.[10,16] According to the Saaty and Vargas's definitions, a prioritisation method is said to preserve rank *weakly* if $a_{ij} \geq 1$ implies $w_i \geq w_j$. The method of solution preserves rank *strongly* if $a_{ik} \geq a_{jk}$ for all $k$ implies $w_i \geq w_j$.

Saaty and Vargas compare the rank preservation properties of the EVM, LLSM and LSM and prove that all three methods preserve rank strongly. In fact the strong rank preservation is a row dominance condition and any acceptable prioritisation method must possess this property.

In our comparison matrix given by (26), the second row $A_2$ dominates the third one $A_3$, that is, $A_2 \succ A_3$ for all values of the parameter $a_{13} \geq 2$. As it can be seen from Tables 3 and 4, all methods give $w_2 > w_3$ for $a_{13} \geq 2$.

When $a_{13} \geq 3$, the first row $A_1$ dominates the second one $A_2$, so we have a strong row dominant matrix, where $A_1 \succ A_2 \succ A_3$. Tables 3 and 4 show that all compared methods preserve rank strongly, since their ranking is $w_1 > w_2 > w_3$ for all values of the parameter $a_{13} \geq 3$.

If the comparison matrix A is consistent, then all prioritisation methods preserve rank weakly, because in the consistent case the inequality $w_i \geq w_j$ implies $a_{ij} \geq 1$. But if the matrix A is inconsistent then the weak rank preservation is not guaranteed.

In their comparative study Golany and Kress[10] showed that the known prioritisation methods do not preserve rank

**Table 2** Euclidean distance of the compared methods

| | *E* | | | | |
|---|---|---|---|---|---|
| *a13* | *FPM* | *EVM* | *LLSM* | *WLSM* | *GPM* |
| 0.01 | 2.341[a] | 3.179 | 3.179 | 3.430 | 5.990[b] |
| 0.1 | 2.299[a] | 2.700 | 2.700 | 3.287 | 5.900[b] |
| 1 | 1.907 | 1.816[a] | 1.816[a] | 2.291 | 5.000[b] |
| 2 | 1.500 | 1.416[a] | 1.416[a] | 1.644 | 4.000[b] |
| 3 | 1.112 | 1.078[a] | 1.078[a] | 1.171 | 1.500[b] |
| 4 | 0.735[a] | 0.737 | 0.737 | 0.762 | 1.000[b] |
| 5 | 0.365[a] | 0.379 | 0.379 | 0.377 | 0.500[b] |
| 6 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 7 | 0.362[a] | 0.399 | 0.399 | 0.378 | 0.500[b] |
| 8 | 0.721[a] | 0.817 | 0.817 | 0.758 | 1.000[b] |
| 9 | 1.079[a] | 1.252 | 1.252 | 1.142 | 1.500[b] |
| 10 | 1.436[a] | 1.703 | 1.703 | 1.531 | 2.000[b] |
| 20 | 4.979[a] | 6.847 | 6.847 | 5.570 | 10.198[b] |
| 100 | 33.246[a] | 61.109[b] | 61.109[b] | 39.561 | 47.000 |

[a]Represents the best solution.
[b]Represents the worst solution.

**Table 3** Priorities derived by the EVM (LLSM) and the FPM

| | *EVM (LLSM)* | | | *FPM* | | |
|---|---|---|---|---|---|---|
| *a13* | $w_1$ | $w_2$ | $w_3$ | $w_1$ | $w_2$ | $w_3$ |
| 0.01 | 0.059 | 0.247 | 0.694 | 0.376 | 0.375 | 0.249 |
| 0.1 | 0.181 | 0.355 | 0.463 | 0.384 | 0.372 | 0.244 |
| 1 | 0.407 | 0.370 | 0.224 | 0.450 | 0.350 | 0.200 |
| 2 | 0.484 | 0.349 | 0.168 | 0.500 | 0.333 | 0.167 |
| 3 | 0.528 | 0.333 | 0.140 | 0.536 | 0.321 | 0.143 |
| 4 | 0.558 | 0.320 | 0.122 | 0.563 | 0.313 | 0.125 |
| 5 | 0.582 | 0.309 | 0.109 | 0.583 | 0.306 | 0.111 |
| 6 | 0.600 | 0.300 | 0.100 | 0.600 | 0.300 | 0.100 |
| 7 | 0.615 | 0.292 | 0.093 | 0.614 | 0.295 | 0.091 |
| 8 | 0.628 | 0.285 | 0.086 | 0.625 | 0.292 | 0.083 |
| 9 | 0.639 | 0.279 | 0.081 | 0.635 | 0.288 | 0.077 |
| 10 | 0.649 | 0.274 | 0.077 | 0.643 | 0.286 | 0.071 |
| 20 | 0.710 | 0.237 | 0.053 | 0.688 | 0.271 | 0.042 |
| 100 | 0.819 | 0.160 | 0.021 | 0.736 | 0.255 | 0.010 |

**Table 4** Priorities derived by the GPM and the WLSM

| | *GPM* | | | *WLSM* | | |
|---|---|---|---|---|---|---|
| *a13* | $w_1$ | $w_2$ | $w_3$ | $w_1$ | $w_2$ | $w_3$ |
| 0.01 | 0.600 | 0.300 | 0.100 | 0.500 | 0.350 | 0.150 |
| 0.1 | 0.600 | 0.300 | 0.100 | 0.499 | 0.348 | 0.153 |
| 1 | 0.600 | 0.300 | 0.100 | 0.500 | 0.333 | 0.167 |
| 2 | 0.600 | 0.300 | 0.100 | 0.520 | 0.320 | 0.160 |
| 3 | 0.545 | 0.273 | 0.182 | 0.544 | 0.311 | 0.144 |
| 4 | 0.571 | 0.286 | 0.143 | 0.567 | 0.306 | 0.128 |
| 5 | 0.588 | 0.294 | 0.118 | 0.585 | 0.302 | 0.113 |
| 6 | 0.600 | 0.300 | 0.100 | 0.600 | 0.300 | 0.100 |
| 7 | 0.609 | 0.304 | 0.087 | 0.612 | 0.299 | 0.089 |
| 8 | 0.615 | 0.308 | 0.077 | 0.622 | 0.298 | 0.080 |
| 9 | 0.621 | 0.310 | 0.069 | 0.630 | 0.297 | 0.073 |
| 10 | 0.625 | 0.313 | 0.063 | 0.636 | 0.297 | 0.067 |
| 20 | 0.625 | 0.313 | 0.063 | 0.668 | 0.297 | 0.035 |
| 100 | 0.662 | 0.331 | 0.007 | 0.694 | 0.299 | 0.007 |

weakly, especially when the degree of inconsistently is greater. This is also shown by our results, given in Tables 3 and 4.

It can be seen that $w_1$ increases and $w_3$ decreases monotonically as $a_{13}$ increases for the EVM (LLSM) and FPM, which however is not valid for both the GPM and the WLSM. Since $a_{12} = 2$ and $a_{23} = 3$, the weak rank preservation condition implies $w_1 > w_2$ and $w_2 > w_3$ for all $a_{13} > 1$. From the results represented in Tables 3 and 4 we can conclude that all compared methods preserve rank weakly for $a_{13} \geq 1$, so that $w_1 > w_2 > w_3$.

When $a_{13} < 1$, the weak rank preservation condition implies $w_3 > w_1$, but since both $a_{12}$ and $a_{23}$ are greater than one we have an obvious contradiction, that is, $w_1 > w_2 > w_3 > w_1$ and it is not possible to find priorities, satisfying simultaneously all weak rank preservation conditions. As it can be seen from Tables 3 and 4, for $a_{13} < 1$ the ranking of the EVM (LLSM) is reversed, that is, $w_3 > w_2 > w_1$. The FPM, GPM and WLSM preserve the same ranking $w_1 > w_2 > w_3$ for all values of the parameter $a_{13}$.

In order to evaluate the weak rank preservation properties of all methods, we can use the Minimum Violations criterion (also called Element Preference Reversal, Golany and Kress[10]). It measures the sum of all violations, occurring when an element $E_i$ is preferred to $E_j$ in the pairwise comparison ($a_{ij} > 1$), but $E_j$ receives a larger weight in the generated priority vector ($w_j > w_i$).

According to this criterion, the ranking $w_3 > w_2 > w_1$, obtained by the EVM (LLSM) for $a_{13} < 1$ has two violations ($w_2 > w_1$ and $w_3 > w_2$, while $a_{12} = 2$and $a_{23} = 3$). The ranking of the FPM, the GPM and the WLSM is $w_1 > w_2 > w_3$ and it has only one violation ($w_1 > w_3$, while $a_{13} < 1$).

The rank preservation is another important property of the proposed FPM, shown by this experiment.

*Example 2. Wealth-of-nations problem*

The Saaty's wealth-of-nations problem[1] has been used by many authors for evaluation of different prioritisation methods. The wealth-of-nations pairwise comparison matrix is given in Table 5. It represents the responses of an economic expert, who compress the wealth of seven countries using the pairwise comparisons within the Saaty's scale $\frac{1}{9}$–9.

The priorities generated by the five compared methods are represented in the upper part of Table 6. It is seen that all methods give different priorities, but identical final rankings of the wealth of the nations. In this case all methods preserve the same rank since the comparison matrix is rather consistent, its consistency index is $\mathrm{CI} = 0.101$ and the consistency ratio is $\mathrm{CR} = 0.077$. The consistency index of the FPM for this problem is $\mu = 0.986$.

The last row of Table 6 represents the values of the Euclidean distance criterion, (25) for all considered prioritisation methods. In this particular problem the best performer is the GPM, being slightly better than the WLSM and FPM, while the most popular prioritisation methods EVM and LLSM give the worst results.

The comparison analysis between the FPM and the other prioritisation methods does not pretend to be a comprehensive one. The main objective of this analysis is just to give an idea of the performance of this new method and to show its features. Presently an extensive analysis has been carried out with randomly generated matrices of higher order, taking into account more evaluation criteria. The preliminary results confirm that the FPM method can be

**Table 5** Wealth-of-nations matrix

| *Country* | *US* | *USSR* | *China* | *France* | *UK* | *Japan* | *W Germany* |
|---|---|---|---|---|---|---|---|
| US | 1 | 4 | 9 | 6 | 6 | 5 | 5 |
| USSR | $\frac{1}{4}$ | 1 | 7 | 5 | 5 | 3 | 4 |
| China | $\frac{1}{9}$ | $\frac{1}{7}$ | 1 | $\frac{1}{5}$ | $\frac{1}{5}$ | $\frac{1}{7}$ | $\frac{1}{5}$ |
| France | $\frac{1}{6}$ | $\frac{1}{5}$ | 5 | 1 | 1 | $\frac{1}{3}$ | $\frac{1}{3}$ |
| UK | $\frac{1}{6}$ | $\frac{1}{5}$ | 5 | 1 | 1 | $\frac{1}{3}$ | $\frac{1}{3}$ |
| Japan | $\frac{1}{5}$ | $\frac{1}{3}$ | 7 | 3 | 3 | 1 | 2 |
| W Germany | $\frac{1}{5}$ | $\frac{1}{4}$ | 5 | 3 | 3 | $\frac{1}{2}$ | 1 |

**Table 6** Comparison of the priorities for the wealth-of-nations matrix

| *Country* | *EVM* | *LLSM* | *WLSM* | *GPM* | *FPM* |
|---|---|---|---|---|---|
| US | 0.427 | 0.417 | 0.487 | 0.373 | 0.501 |
| USSR | 0.230 | 0.231 | 0.175 | 0.290 | 0.160 |
| China | 0.021 | 0.020 | 0.030 | 0.041 | 0.043 |
| France | 0.052 | 0.054 | 0.059 | 0.062 | 0.060 |
| UK | 0.052 | 0.054 | 0.059 | 0.062 | 0.060 |
| Japan | 0.123 | 0.128 | 0.128 | 0.096 | 0.1 |
| W Germany | 0.094 | 0.096 | 0.096 | 0.075 | 0.075 |
| Euclidean distance E | 13.708 | 14.206 | 10.381 | 8.847 | 10.888 |

used as a worthy alternative of the existing prioritisation methods in the AHP.

## Conclusions

A new fuzzy programming method for deriving problems in the AHP is proposed, based on fuzzy geometrical representations of the prioritisation problem. The FPM is as an optimisation method for prioritisation, since it derives priorities that maximise the degree of intersection between the fuzzy lines, approximately describing the human preferences. This method transforms the prioritisation problem into a standard linear program.

Incomprehensive numerical comparison with other known methods for priority assessment is performed to justify the properties of the FPM. The comparison analysis shows that the FPM overperforms some of the existing methods, especially in highly inconsistent cases.

The new method has some attractive properties, such as a natural consistency index, simplicity of the computation algorithm, good rank preservation and precision and can be used as an alternative to the known prioritisation methods. It can be modified and applied for interval comparisons and group decision making, which are the directions of our future research.

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*Received February 1999;*
*accepted October 1999 after one revision*